

## PAUL ERDŐS AND ADDITIVE BASES

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ABSTRACT. This is a survey of some of Erdős's work on bases in additive number theory.

## 1. ADDITIVE BASES

Paul Erdős, while he was still in his 20s, wrote a series of extraordinarily beautiful papers in additive and combinatorial number theory. The key concept is *additive basis*.

Let  $A$  be a set of nonnegative integers, let  $h$  be a positive integer, and let  $hA$  denote the set of integers that can be represented as the sum of exactly  $h$  elements of  $A$ , with repetitions allowed. A central problem in additive number theory is to describe the sumset  $hA$ . The set  $A$  is called an *additive basis of order  $h$*  if every nonnegative integer can be represented as the sum of exactly  $h$  elements of  $A$ . For example, the set of squares is a basis of order 4 (Lagrange's theorem) and the set of nonnegative cubes is a basis of order 9 (Wieferich's theorem).

The set  $A$  of nonnegative integers is an *asymptotic basis of order  $h$*  if  $hA$  contains every sufficiently large integer. For example, the set of squares is an asymptotic basis of order 4 but not of order 3. The set of nonnegative cubes is an asymptotic basis of order at most 7 (Linnik's theorem), and, by considering congruences modulo 9, an asymptotic basis of order at least 4. The Goldbach conjecture implies that the set of primes is an asymptotic basis of order 3.

The modern theory of additive number theory begins with the work of Lev Genrikhovich Shnirel'man (1905-1938). In an extraordinary paper [37], published in Russian in 1930 and republished, in an expanded form [38], in German in 1933, he proved that every sufficiently large integer is the sum of a bounded number of primes. Not only did Shnirel'man apply the Brun sieve, which Erdős subsequently developed into one of the most powerful tools in number theory, but he introduced a new density for a set of integers that is exactly the right density for the investigation of additive bases. (For a survey of the classical bases in additive number theory, see Nathanson [27].)

## 2. SHNIREL'MAN DENSITY AND ESSENTIAL COMPONENTS

The *counting function*  $A(x)$  of a set  $A$  of nonnegative integers counts the number of positive integers in  $A$  that do not exceed  $x$ , that is,

$$A(x) = \sum_{\substack{a \in A \\ 1 \leq a \leq x}} 1.$$

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The *Shnirel'man density* of  $A$  is

$$\sigma(A) = \inf_{n=1,2,\dots} \frac{A(n)}{n}.$$

The sum of the sets  $A$  and  $B$  is the set  $A + B = \{a + b : a \in A \text{ and } b \in B\}$ . Shnirel'man proved the fundamental sumset inequality:

$$\sigma(A + B) \geq \sigma(A) + \sigma(B) - \sigma(A)\sigma(B).$$

This implies that if  $\sigma(A) > 0$ , then  $A$  is a basis of order  $h$  for some  $h$ . This does not apply directly to the sets of  $k$ th powers and the set of primes, which have Shnirel'man density 0. However, it is straightforward that if  $\sigma(A) = 0$  but  $\sigma(h'A) > 0$  for some  $h'$ , then  $A$  is a basis of order  $h$  for some  $h$ .

Landau conjectured the following strengthening of Shnirel'man's addition theorem, which was proved by Mann [22] in 1942:

$$\sigma(A + B) \geq \sigma(A) + \sigma(B).$$

Artin and Scherk [1] published a variant of Mann's proof, and Dyson [4], while an undergraduate at Cambridge, generalized Mann's inequality to  $h$ -fold sums. Nathanson [26] and Hegedüs, Piroška, and Ruzsa [17] have constructed examples to show that the Shnirel'man density theorems of Mann and Dyson are best possible.

We define the *lower asymptotic density* of a set  $A$  of nonnegative integers as follows:

$$d_L(A) = \liminf_{n=1,2,\dots} \frac{A(n)}{n}.$$

This is a more natural density than Shnirel'man density. A set  $A$  with asymptotic density  $d_L(A) = 0$  has Shnirel'man density  $\sigma(A) = 0$ , but not conversely. A set  $A$  with asymptotic density  $d_L(A) > 0$  is not necessarily an asymptotic basis of finite order, but  $A$  is an asymptotic basis if  $d_L(A) > 0$  and  $\gcd(A) = 1$  (cf. Nash and Nathanson [23]).

The set  $B$  of nonnegative integers is called an *essential component* if

$$\sigma(A + B) > \sigma(A)$$

for every set  $A$  such that  $0 < \sigma(A) < 1$ . Shnirel'man's inequality implies that every set of positive Shnirel'man density is an essential component. Of course, there exist very sparse sets of zero asymptotic density that are not essential components. Khinchin [19] proved that the set of nonnegative squares is an essential component. Note that the set of squares is a basis of order 4. Using an extremely clever elementary argument, Erdős [6], at the age of 22, proved the following considerable improvement: Every additive basis is an essential component. Greatly impressed, Landau celebrated this result in his 1937 Cambridge Tract *Über einige neuere Fortschritte der additiven Zahlentheorie* [21].

Plünnecke [32, 33, 34] and Ruzsa [36] have made important contributions to the study of essential components.

### 3. THE ERDŐS-TURÁN CONJECTURE

In another classic paper, published in 1941, P. Erdős and P. Turán [5] investigated Sidon sets. The set  $A$  of nonnegative integers is a *Sidon set* if every integer has at most one representation as the sum of two elements of  $A$ . They concluded their paper as follows:

Let  $f(n)$  denote the number of representations of  $n$  as  $a_i + a_j$ ,  
 .... If  $f(n) > 0$  for  $n > n_0$ , then  $\limsup f(n) = \infty$ . Here we  
 may mention that the corresponding result for  $g(n)$ , the number  
 of representations of  $n$  as  $a_i a_j$ , can be proved.

The additive statement is still a mystery. The Erdős-Turán conjecture, that the representation function of an asymptotic basis of order 2 is always unbounded, is a major unsolved problem in additive number theory.

Many years later, in 1964, Erdős [7] published the proof of the multiplicative statement. This proof was later simplified by Nešetřil and Rödl [31], and generalized by Nathanson [25].

Long ago, while a graduate student, I searched for a counterexample to the Erdős-Turán conjecture. Such a counterexample might be extremal in several ways. It might be “thin” in the sense that it contains few elements. Every asymptotic basis of order  $h$  has counting function  $A(x) \gg x^{1/h}$ . We call an additive basis of order  $h$  *thin* if  $A(x) \ll x^{1/h}$ . Thin bases exist. The first examples were constructed in the 1930s by Raikov [35] and by Stöhr [39], and Cassels [2, 28] later produced another important class of examples.

Alternatively, an asymptotic basis  $A$  of order  $h$  might be extremal in the sense that no proper subset of  $A$  is an asymptotic basis of order  $h$ . This means that removing *any* element of  $A$  destroys every representation of infinitely many integers. It is not obvious that minimal asymptotic bases exist, but I was able to construct asymptotic bases of order 2 that were both thin and minimal. Of course, none was a counterexample to the Erdős-Turán conjecture.

Stöhr [40] gave the first definition of minimal asymptotic basis, and Härtter [16] gave a non-constructive proof that there exist uncountably many minimal asymptotic bases of order  $h$  for every  $h \geq 2$ .

There is a natural dual to the concept of a minimal asymptotic basis. We call a set  $A$  an *asymptotic nonbasis of order  $h$*  if it is not an asymptotic basis of order  $h$ , that is, if there are infinitely many positive integers not contained in the sumset  $hA$ . An asymptotic nonbasis of order  $h$  is *maximal* if  $A \cup \{b\}$  is an asymptotic basis of order  $h$  for every nonnegative integer  $b \notin A$ . The set of even nonnegative integers is a trivial example of a maximal nonbasis of order  $h$  for every  $h \geq 2$ , and one can construct many other examples that are unions of the nonnegative parts of congruence classes. It is difficult to construct nontrivial examples.

I discussed this and other open problems in my first paper [24] in additive number theory. I did not know Erdős at the time, but I mailed him a preprint of the article. It still amazes me that he actually read this paper sent to him out of the blue by a completely obscure student, and he answered with a long letter in which he discussed his ideas about one of the problems. This led to correspondence, meetings, and joint work over several decades.

#### 4. EXTREMAL PROPERTIES OF BASES

Here is a small sample of results on minimal bases and maximal nonbases.

Nathanson and Sarközy [30] proved that if  $A$  is a minimal asymptotic basis of order  $h \geq 2$ , then  $d_L(A) \leq 1/h$ . The proof uses Kneser’s theorem [20] on the asymptotic density of sumsets, one of the most beautiful and most forgotten theorems in additive number theory. A well known special case is Kneser’s theorem for the sum of finite subsets of a finite abelian group.

Erdős and Nathanson [14] proved that, for every  $h \geq 2$ , there exist minimal asymptotic bases of order  $h$  with asymptotic density  $1/h$ . Moreover, for every  $\alpha \in (0, 1/(2h-2))$ , there exist minimal asymptotic bases of order  $h$  with asymptotic density  $\alpha$ . In particular, for every  $\alpha \in (0, 1/2]$ , there exist minimal asymptotic bases of order 2 with asymptotic density  $\alpha$ .

Does every asymptotic basis  $A$  of order 2 contain a minimal asymptotic basis of order 2? Sometimes. Let  $f(n)$  count the number of representations of  $n$  as the sum of two elements of  $A$ . If  $f(n) > c \log n$  for some  $c > (\log(4/3))^{-1}$  and all sufficiently large  $n$ , then  $A$  contains a minimal asymptotic basis of order 2 (Erdős-Nathanson [13]). This result is almost certainly not best possible.

Does every asymptotic basis of order 2 contain a minimal asymptotic basis of order 2? No. There exists an asymptotic basis  $A$  of order 2 with the following property: If  $S \subseteq A$ , then  $A \setminus S$  is an asymptotic basis of order 2 if and only if  $S$  is finite (Erdős-Nathanson [12]).

There exist “trivial” maximal asymptotic nonbases of order  $h$  consisting of unions of arithmetic progressions [24]. However, for every  $h \geq 2$ , there also exist nontrivial maximal asymptotic nonbases of order  $h$  (Erdős-Nathanson [8, 11] and Deshouillers and Grekos [3]).

Is every asymptotic nonbasis of order  $h$  a subset of a maximal asymptotic nonbasis of order  $h$ ? Sometimes. If  $A \cup S$  is an asymptotic nonbasis of order 2 for every finite set  $S \subseteq \mathbf{N} \setminus A$ , then  $A$  is contained a maximal asymptotic nonbasis of order 2.

Is every asymptotic basis of order  $h$  a subset of a maximal asymptotic nonbasis of order  $h$ ? No. Hennefeld [18] proved that, for every  $h \geq 2$  there exists an asymptotic nonbasis  $A$  of order  $h$  such that, if  $S \subseteq \mathbf{N} \setminus A$ , then  $A \cup S$  is an asymptotic nonbasis  $A$  of order  $h$  if and only if the set  $\mathbf{N} \setminus (A \cup S)$  is infinite.

Investigating extremal properties of additive bases is like exploring for new plant species in the Amazon rain forest. Much has been collected, but much more is unimagined. The following results about oscillations of bases and nonbases appear in [9, 10].

There exists a minimal asymptotic basis of order 2 such that  $A \setminus \{x\}$  is a maximal asymptotic nonbasis of order 2 for every  $x \in A$ .

There exists a maximal asymptotic nonbasis of order 2 such that  $A \cup \{y\}$  is a minimal asymptotic basis of order 2 for every  $y \in \mathbf{N} \setminus A$ .

There exists a partition of the nonnegative integers into disjoint sets  $A$  and  $B$  such that  $A$  is a minimal asymptotic basis of order 2 and  $B$  is a maximal asymptotic nonbasis of order 2.

There exists a partition of the nonnegative integers into disjoint sets  $A$  and  $B$  that oscillate in phase from minimal asymptotic basis of order 2 to maximal asymptotic nonbasis of order 2 as random elements are moved from the basis to the nonbasis, infinitely often.

It is an open problem to extend these results to asymptotic bases of order  $h \geq 3$ . For a survey of external problems in additive number theory, see Nathanson [29].

Bases fascinated Erdős, but the subject is still obscure, an obsession of the few. I wrote my first paper [24] on minimal bases and maximal nonbases in additive number theory in the summer of 1970, while a graduate student, visiting the Weizmann Institute in Israel. I was thinking about the Erdős-Turán conjecture, and studied the foundational book *Sequences* [15] by Halberstam and Roth (now, unfortunately,

out of print). Returning to Weizmann to lecture in 2001, I looked for the book in the library. In the ancient, pre-computer era, a library book had a  $3 \times 5$  card. When you checked out a book, you wrote your name on the card. You could see who had read any book. After 30 years, mine was still the only name on the card for *Sequences*.

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